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A NOTE ON CHAIN TRANSITIVITY OF LINEAR DYNAMICAL SYSTEMS

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Abstract. In this paper we study some topological modes of recurrent sets of linear homeomorphisms of a finite-dimensional topological vector space. More precisely, we show that there are no chain transitive linear homeomorphisms of a finite-dimensional Banach space having the shadowing property. Then, we give examples to illustrate our results.

1. Introduction and preliminaries

Several topological modes of recurrence(e.g., chain recurrence, chain transitivity and topological transivity) described by the eventual behavior of the orbits of a dynamical system have an important role to understand the qualitative theory of the dynamical system. The chain recurrence due to Conley [5] is an extended notion of topological transitivity by using ε-chains. Shimomura [12] studied the chain transitive sets of discrete dynamical systems from the view point of topological entropy. Crovisier [6] proved that the chain transitive sets of C^1-generic diffeomorphisms are approximated in the Hausdorff topology by periodic orbits. Also Sakai [11] studied the chain transitive sets in view of shadowing theory and showed that the chain transitive sets are natural candidates to replace hyperbolic basic sets in Smale's hyperbolic theory for dynamical systems. Furthermore, Morimoto [7] characterized linear automorphisms of $\mathbb{R}^n$ be the shadowing property, i.e., the author showed that various notions of hyerbolicity, expansivity, structural stability, shadowing property and topological stability are equivalent for linear automorphisms on $\mathbb{R}^n$.

To state our results, let us recall some basic notions of dynamical systems which are used in the sequel(see [3, 9]). Let X be a metric space with metric d and $f : X \to X$ be a homeomorphism.

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For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ in X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called to be *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for each $i \in \mathbb{Z}$. We say that a homeomorphism $f : X \to X$ has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f is ε-shadowed by some point in X. The theory related to various types of shadowing has been widely studied in the class of dynamical systems on a compact metric space(see [1, 8]).

We say that a homeomorphism $f : X \to X$ is called *topologically transitive* if for every pair of non-empty open subsets U and V of X, there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Then we note that f is topologically transitive if and only if there is a point $x \in X$ such that the orbit $Orb_f(x)$ of $x \in X$ is dense in X when X is a complete separable metric space, where $Orb_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$. It is well known that topological transitivity is preserved under a topological conjugacy. We say that a homeomorphism $f : X \to X$ of a metric space X is called *chain transitive* if for every $\varepsilon > 0$ and any pair x, y of points of X, there exists a finite ε-chain $\{x_i\}_{i=0}^n$ of points of X such that $x_0 = x$ and $x_n = y$, where $\{x_i\}_{i=0}^n$ is an *ε-chain* from x to y if $d(f(x_i), x_{i+1}) < \varepsilon$ for each $i = 0, \cdots, n-1$. We note that if f is topologically transitive, then f is chain transitive and the converse does not hold in general. If f has the shadowing property, then topological transitivity coincides with chain transitivity.

We say that a point x of X is called *nonwandering* of f if for every neighborhood U of x, there is a non-zero $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. Let $\Omega(f)$ denote the set of nonwandering points of f, which is called the *nonwandering* set of f. We say that a homeomorphism $f : X \to X$ is called *nonwandering* if $\Omega(f) = X$. We say that a point x of X is called *chain recurrent* of f if for every $\varepsilon > 0$ there is a finite ε-chain $\{x_i\}_{i=0}^n$ such that $x_0 = x_n = x$. Let $CR(f)$ denote the set of chain recurrent points of f, which is called the *chain recurrent* set of f. We say that a homeomorphism $f : X \to X$ is called *chain recurrent* if for any $\varepsilon > 0$ and any point $x \in X$, there is a finite ε-pseudo-chain $\{x_i\}_{i=0}^n$ such that $x_0 = x_n = x$, i.e., $CR(f) = X$. We note that every nonwandering homeomorphism is chain recurrent, and every chain recurrent homeomorphism with the shadowing property is nonwandering(see [2]). Also, we note that the chain transitivity implies the chain recurrence.

We say that a linear homeomorphism $f : X \to X$ of a topological vector space(or a Banach space) X is called *hypercyclic* if there is a nonzero vector $x \in X$ such that the orbit $Orb_f(x)$ is dense in X. The hypercyclicity means the topological transitivity when X is a separable complete metric space(see [4, pp. 2 and 26], [3, Theorem 1.2]).

In this paper we study some relations between chain transitivity and chain recurrence in the class of linear homeomorphisms on a finite-dimensional Banach space. Also, we give some examples to illustrate our results. More precisely, we state our main result.

THEOREM 1.1. *There are no chain transitive linear homeomorphisms of a finite-dimensional Banach space with the shadowing property.*

2. Proof of Theorem 1.1

In this section we study relations with various modes of recurrence sets for linear homeomorphisms on a finite-dimensional Banach space. Then we give a proof Theorem 1.1 and examples to our results.

LEMMA 2.1. *Let X be a metric space and $f : X \to X$ be a homeomorphism with the shadowing property. If f is chain transitive, then it is topologically transitive.*

Proof. Let U and V be non-empty open subsets of a metric space X with metric d. We claim that there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Choose $r > 0$ such that $B_d(x, r) \subset U$ and $B_d(y, r) \subset V$ for some $x \in U, y \in V$, where $B_d(a, r) = \{b \in X \mid d(a, b) < r\}$. Let $\delta > 0$ be corresponding to $r > 0$ by the shadowing property. Since f is chain transitive, there is a finite δ-chain $\{x_i\}_{i=0}^n$ such that $x_0 = x, x_n = y$. Since f has the shadowing property, there exists $z \in X$ such that $d(x, z) < r$ and $d(f^n(z), y) < r$. Hence, $f^n(z) \in f^n(U) \cap V$. □

The proof of the following result was given in [3, Proposition 1.1]. For completion of our main result, we again give a detailed proof.

LEMMA 2.2. *Any linear homeomorphism of a finite-dimensional Banach space is not hypercyclic.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be any linear homeomorphism on $\mathbb{R}^n$. Suppose by contrary that f is hypercyclic and pick a hypercyclic vector $x \in \mathbb{R}^n$. Then we observe that $\{x, f(x), \ldots, f^{n-1}(x)\}$ is a linearly independent family and hence it is a basis of $\mathbb{R}^n$. Then we see $f^m(x) \in < x, f(x), \ldots, f^i(x) >$ for every $m \in \mathbb{N}$. If follows that

$$\overline{\{f^m(x)\}_{m\in\mathbb{N}}} \subset < x, f(x), \ldots, f^i(x) > \neq \mathbb{R}^n,$$

where $\overline{A}$ denotes the closure of a set A and $< x >= \{cx \mid c \in \mathbb{R}\}$. Therefore $\{x, f(x), \ldots, f^{i+1}(x)\}$ is linearly independent for all $i < n-1$.

If $\{f^m(x) \mid m \in \mathbb{N}\}$ is dense, then for any $\alpha \in \mathbb{R}$ there is a subsequence $\{n_k\}$ such that $f^{n_k}(x) \to \alpha x$ as $k \to \infty$. On the other hand, by hypercyclic of f, for any $y \in \mathbb{R}^n$ there is a subsequence $\{m_l\} \subset \mathbb{N}$ such that $f^{m_l}(x) \to y$ as $l \to \infty$. Then we have that

$$\lim_{k\to\infty} f^{n_k}(y) = \lim_{k,l\to\infty} f^{m_l} f^{n_k}(x) = \lim_{l\to\infty} f^{m_l}\alpha(x) = \alpha y$$

for every $y \in \mathbb{R}^n$. Then $f^{n_k} \to \alpha I$ as $k \to \infty$ for every $\alpha \in \mathbb{R}$. By the continuity of the determinant, we have $\det(f^{n_k}) \to \det(\alpha I) = \alpha^n$ as $k \to \infty$. Since $\det(f^{n_k}) = (\det(f))^{n_k}$, for every $\beta \in \mathbb{R}$ there exists a subsequence $\{n_k\}$ such that $(\det(f))^{n_k} \to \beta = \alpha^n$ as $k \to \infty$. Therefore we get the contradiction that

$$\overline{\{(\det(f))^m \mid m \in \mathbb{N}\}} = \mathbb{R}.$$

This completes the proof. □

We give an example to explain that there are hypercyclic operators on an infinite-dimensional Banach space.

EXAMPLE 2.3. *[10, Theorem 1] Let X be either $l^p(1 \le p < \infty)$ or c_0. For any arbitrary $a > 1$, there is a linear map $f : X \to X$ and a point x_0 of X such that $\mathrm{Orb}_f(x_0)$ is dense in X.*

From Example 2.3, we note that hypercyclicity turns out to be a pure infinite-dimensional phenomenon(see [3, Example 1.9]).

LEMMA 2.4. *Let f, g be linear homeomorphisms on an infinite-dimensional Banach space X. If there exists a homeomorphism g such that $fh = hg$ then f is hypercyclic if and only if g is hypercyclic.*

Proof. Suppose that f is hypercylic. Since h is bijective, $h^{-1}f = gh^{-1}$. Let $x \in X$ be a non-zero vector such that $\overline{\{f^n(x)\}} = X$. We claim that for any $y \in X$ there is a subsequence $\{n_k\}$ such that $f^{n_k}(h^{-1}(x)) \to y$ as $k \to \infty$. Let $z = h(y)$ for every $y \in X$. Since f is hypercyclic, there exists a subsequence $\{n_k\}$ such that $f^{n_k}x \to z$ as $k \to \infty$. Then we have that

$$g^{n_k}(h^{-1}(x)) = h^{-1}(f^{n_k}(x)) \to h^{-1}(z) = y \text{ as } k \to \infty.$$

Hence f is hypercyclic. Similarly, we can prove the converse. This completes the proof. □

Next we give an example to explain relations between two notions of the chain transitivity and the shadowing property of a linear homeomorphism on $\mathbb{R}^n$.

EXAMPLE 2.5. *(1) The identity linear homeomorphism $I_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ (or isometry) is chain transitive, but $I_{\mathbb{R}^n}$ does not have the shadowing property.*

(2) The linear homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x) = \frac{1}{2}x$ is not chain transitive but has the shadowing property.

Proof. Let $B(x,r) = \{y \in \mathbb{R}^2 \mid ||x-y|| < r\}$ be an open ball of x with radius $r > 0$. For every $n \geq 1$, $f^n(B(\mathbf{0},1)) = B(\mathbf{0}, \frac{1}{2^n})$. Take $x = (1,0) \in \mathbb{R}^2$. Suppose that $\{x = x_0, x_1, \ldots, x_n = x)\}$ is a $\frac{1}{4}$-chain from x to itself. From $f(x_0) = (\frac{1}{2}, 0) \in B(\mathbf{0}, \frac{1}{2})$, we have that $x_1 \in B(\mathbf{0}, 1)$ and $x_1 \neq x$. Using induction, we have that $x_n \in B(\mathbf{0}, 1)$ and $x_n \neq x$ for all $n \geq 1$. Hence f is not chain transitive. □

PROPOSITION 2.6. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear homeomorphism satisfying $||f|| < 1$, then f is not chain transitive. Here $||f|| = \sup\{\frac{||f(x)||}{||x|||} \mid 0 < ||x|| \leq 1\}$ is the norm of f.*

Proof. Suppose $||f|| < 1$ and f is chain transitive. Take $x = (1, 0, \ldots, 0)$. Choose $\epsilon = (1 - ||f||)/2$ and consider an ϵ-chain $\{x = x_0, x_1, \ldots, x_n = x\}$ from x to itself. Since $||f(x_0) - \mathbf{0}|| < ||f||$ and $||x_1 - \mathbf{0}|| < \epsilon + ||f|| < 1$, then $x_1 \in B(\mathbf{0}, \frac{1+||f||}{2})$. Similarly, since $||x_2 - f(x_1)|| < \frac{1-||f||}{2}$, we have $||x_2 - \mathbf{0}|| < 1$. Continuing in this way, we get the conclusion that if $||x_i|| < \frac{1}{2} + \frac{||f||}{2} < 1$, then $||x_{i+1}|| < \frac{1}{2} + \frac{||f||}{2} < 1$ and $x_i \neq x$ for all $i \in \mathbb{N}$. This is absurd. □

PROPOSITION 2.7. *Any linear homeomorphism on a finite-dimensional Banach space is chain transitive if and only if it is chain recurrent.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be any linear homeomorphism. Then it is easy to see that the chain transitivity implies the chain recurrence. Thus we claim that the chain recurrence of f implies the chain transitivity. Let

$$\gamma_k = \{x_k = x_0^{(k)}, x_1^{(k)}, \ldots, x_{m_k}^{(k)}, x_{m_k+1}^{(k)} = x_k\}$$

be an ϵ-chain from x_k to itself for each k with $0 \leq k \leq n-1$.

For $k = 0$, we have that

$$||f(x_{m_0}^{(0)}) - x_0^{(1)}|| \leq ||f(x_{m_0}^{(0)}) - x_{m_0+1}^{(0)}|| + ||x_{m_0+1}^{(0)} - x_0^{(1)}|| < \epsilon + 2\epsilon = 3\epsilon.$$

Continuing in this way, there exists a 3ϵ-chain from x to y of f (see Figure 1). Since ϵ is arbitrary small, f is chain transitive. □

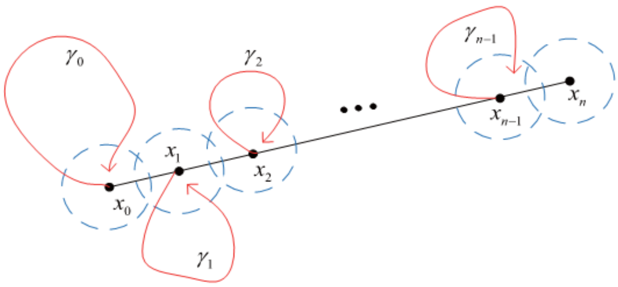

Figure 1. A 3ϵ-chain from x to y of f

Remark 2.8. C^0-generically, there are linear homeomorphisms f of $\mathbb{R}^n$ such that the chain transitivity of f is not equivalent to chain recurrence. Denote by $\mathrm{SP}(\mathbb{R}^n)$ the set of (or hyperbolic) linear homeomorphisms with the shadowing property. Then we see that $\mathrm{SP}(\mathbb{R}^n)$ is both open and dense in $LH(\mathbb{R}^n)$ and $\mathrm{SP}(\mathbb{R}^n) \cap \mathrm{CT}(\mathbb{R}^n) = \emptyset$. Here $LH(\mathbb{R}^n)$ is the space of all liner homeomorphisms on $\mathbb{R}^n$.

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear homeomorphism on $\mathbb{R}^n$. Suppose that there is a chain transitive liner homeomorphism f of $\mathbb{R}^n$ with the shadowing property. Then T is hypercyclic by Lemma 2.1. This contradicts the fact that any linear homeomorphism of $\mathbb{R}^n$ is not hypercyclic by Lemma 2.2. This completes the proof. □

From Theorem 1.1, we can easily obtain the following result.

Corollary 2.9. *There are no nonwandering linear homeomorphisms on a finite-dimensional Banach space with the shadowing property.*

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References

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems*, Elsevier science B. V., 1994.

[2] A. Arbieto and E. Rego, Positive entropy through pointwise dynamics, *Proc. Amer. Math. Soc.*, **148** (2020), 263-271.
[3] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, New York, 2009.
[4] G. D. Birkhoff, Surface transformations and their dynamical applications, *Acta Math.*, **43** (1922), 1-119.
[5] C. Conley, *Isolated Invariant Sets and Morse Index*, CBMS Regional Conferences Series in Mathematics., **38**, American Mathematical Society, Providence, R.I., 1978.
[6] S. Crovisier, *Periodic orbits and chain-transitive sets of C^1-diffeomorphims*, Publ. Math. Ins. Hautes Études Sci., **104** (2006), 87-141.
[7] A. Morimoto, *Stochastically stable diffeomorphisms and Takens conjecture*, Surikais Kokyuruko, **303** (1977), 8-24.
[8] S. Y. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Mathematics, **1706**, Springer-Verlag, Berlin, 1999.
[9] S. Y. Pilyugin and K. Sakai, *Shadowing and Hyperbolicity*, Lecture Notes in Mathematics, **2193**, Springer, 2017.
[10] S. Rolewiz, *On orbits of elements*, Studia Math., **32** (1969), 17-22.
[11] K. Sakai, *Shadowable chain transitive sets*, J. Difference. Equ. Appl., **19** (2013), 1601-1618.
[12] T. Shimomura, *On a structure of discrete dynamical systems from the viewpoint of chain components and some applications*, Japan J. Math., **15** (1989), 99-126.

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